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Exact wall solutions in 5-dimensional SUSY QED at finite coupling

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Abstract: A series of exact BPS solutions are found for single and double domain walls in $\mathcal{N}=2$ supersymmetric (SUSY) QED for finite gauge coupling constants. Vector fields are found to be massive, although it is localized on the wall. Massless modes can be assembled into a chiral scalar multiplet of the preserved $\mathcal{N}=1$ SUSY, after an appropriate gauge choice. The low-energy effective lagrangian for the massless fields is obtained for the finite gauge coupling. The inter-wall force is found to be much stronger than the known infinite coupling case. The previously proposed expansion in inverse powers of the gauge coupling has pathological oscillations, and does not converge to the correct finite coupling result.

Keywords: Field Theories in Higher Dimensions, Extra Large Dimensions, Spontaneous Symmetry Breaking, Extended Supersymmetry.

Contents

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1. Introduction

In recent years, much efforts have been devoted to the brane world scenario [1]–[3], where our world is supposed to be realized on a subspace such as a wall embedded in higher dimensional spacetime. On the other hand, supersymmetry (SUSY) has been one of the most fruitful ideas to build realistic unified models beyond the standard model [4]. SUSY helps to obtain domain walls and other stable solutions by requiring a part of SUSY to be preserved by the field configuration. Then the configuration has minimum energy with the given boundary condition and is automatically a solution of field equations. Wall configurations can preserve half of SUSY charges [5]–[12], which are called $\frac{1}{2}$ BPS states [13]. To obtain a wall with the four-dimensional world volume, we need to consider a fundamental theory in five or more spacetime dimensions. If such a theory are supersymmetric, it must have at least eight SUSY charges [14]. Simplest theories with eight SUSY consist of hypermultiplets. The nontrivial interactions of hypermultiplets require either nonlinearity of kinetic term (nonlinear sigma model) or gauge interactions [15]. The target spaces of such nonlinear sigma models are hyper-Kähler (HK) manifolds [16, 17] and can most conveniently be obtained by using vector multiplets as Lagrange multiplier fields [18, 19]. By introducing the Fayet-Iliopoulos (FI) term for the U(1) Lagrange multiplier vector multiplet and mass terms for hypermultiplets, one can obtain nontrivial potential terms with N discrete SUSY vacua for N hypermultiplets. Domain wall solutions have been obtained in these massive nonlinear sigma models with the target space $T^*\mathbf{C}P^{N-1}$ [20]–[26] and their generalizations [27]. One of the most convenient methods to obtain these nonlinear sigma models is to take a limit of infinite gauge coupling from the usual minimal gauge interactions of hypermultiplets, namely massive $\mathcal{N} = 2$ SUSY QED. However, it has been known that the massive $\mathcal{N} = 2$ SUSY QED itself has discrete SUSY vacua and may allow domain walls even with finite gauge coupling [28]–[32].

The massive nonlinear sigma models with multi-flavor have been found to give BPS multiple wall solutions containing a number of parameters, which represent the relative positions and phases of these multiple walls and are called moduli [28]. When these moduli parameters are promoted to functions of the world volume coordinates of the walls, one can obtain the low-energy effective lagrangian of moduli fields, which are massless [33, 34, 22]. It has been noted that gauge field should acquire nontrivial configuration when these moduli depend on the world volume coordinates [30]. It has also been proposed that an expansion in inverse powers of gauge coupling may be useful to obtain the moduli dynamics at finite gauge coupling [28].

To fulfill the goals of the brane-world scenario, we need to localize particles of the standard model to "branes", such as domain walls. It has been a notoriously difficult problem to obtain massless localized gauge field on a wall [2, 35, 36]. The general arguments [2, 35] and an explicit model [36] suggest that gauge fields may be localized on a wall, but will not be massless in the massive $\mathcal{N} = 2$ SUSY QED with finite gauge coupling. Nevertheless, it is still interesting to study explicitly the role and the properties of dynamical gauge fields in the interacting system of the vector multiplet and hypermultiplets. In this respect, it should be useful to obtain exact solutions with finite gauge couplings [32].[1] In the four-dimensional massive $\mathcal{N} = 2$ SUSY QED which can be obtained by a dimensional reduction of our model, it has been argued that a massless gauge field is obtained by dualizing the massless Nambu-Goldstone scalar [30].[2] While this is an intriguing result, the gauge field as a dual of a compact scalar is an intrinsically three-dimensional peculiarity which is not valid to the more realistic situation such as our five-dimensional theory. In another paper, we show that massless localized gauge fields can be obtained by coupling our model to a tensor multiplet [37].

The purpose of our paper is to work out exact solutions of BPS single and multiple walls with finite gauge coupling in massive $\mathcal{N} = 2$ SUSY QED in five dimensions and to study the fluctuations on the domain wall background. We obtain a series of exact solutions for single as well as multiple walls provided gauge coupling has particular values relative to the ratio of the mass parameters and FI parameters. We also show that the gauge multiplet is massive for any finite gauge coupling, irrespective of the possible existence of exact solutions. This is consistent with the previous general arguments [2, 35] and explicit examples [36]. In our model, the massless Nambu-Goldstone scalar still exist, but are apparently unrelated to the dual gauge field. Instead, we obtain the Nambu-Goldstone scalars and the Nambu-Goldstone fermions which form a chiral scalar multiplet of the remaining $\mathcal{N} = 1$ SUSY, by choosing an appropriate gauge. Moreover we have only massive spectra for the vector field. These results are in contrast to the previous results of the massive $\mathcal{N} = 2$ SUSY U(1) gauge theory in four dimensions, where the Nambu-

[1] In ref. [32], an exact solution of domain wall junction has been obtained for the massive $\mathcal{N} = 2$ SUSY QED with the *finite gauge coupling* for the first time. Therefore it automatically implies an exact solution of domain wall for finite gauge coupling.

[2] They are interested in an $\mathcal{N} = 1$ gauge theory obtained from $\mathcal{N} = 2$ SU(2) gauge theory deformed by an adjoint scalar mass term. However, they observed the model reduces to our $\mathcal{N} = 2$ SUSY QED to the leading order of the mass deformation parameter.

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Goldstone scalar is identified as the dual of the three-dimensional gauge field [30]. As a result of exact solutions, we also obtain the low-energy effective lagrangian for massless moduli fields for finite gauge couplings, by promoting the moduli parameters into four-dimensional massless fields [33, 34, 22]. In the process of carrying out the program, we observe that the gauge field should acquire nontrivial configuration with nonvanishing field strength around the wall, confirming the observation in ref. [30]. Moreover, we find that this nontrivial gauge field configuration should be determined by means of the field equation of gauge field. We compute also some lower order terms of the previously proposed expansion in inverse powers of gauge coupling [28], and compare the expansion with our exact result. We find that the expansion oscillates wildly and does not converge to the exact result in any smooth or uniform way. Therefore the expansion has a difficulty or at least subtlety in extracting physical quantities.

In section 2, we introduce our model of massive $\mathcal{N}=2$ SUSY QED and obtain its SUSY vacua and BPS equations. In section 3, we study the single wall in the case of two hypermultiplets ($N=2$), and work out fluctuation spectra to show that there are massless Nambu-Goldstone modes, but that the vector field has only massive fluctuations. In section 4, a series of exact solutions are constructed for $N-1$ walls in the case of N hypermultiplets and obtain the low-energy effective lagrangian for the massless fields. We also work out a few lower order terms of the expansion in inverse powers of gauge coupling and compare it to our exact solution.

2. Massive $\mathcal{N}=2$ SUSY QED and BPS equations

The simplest building block of $\mathcal{N}=2$ SUSY theory in five dimensions is hypermultiplets which consist of $\mathrm{SU}(2)_R$ doublets of complex scalar fields H^{iA}, Dirac fields ψ^A and complex auxiliary field F_i^A, where $i=1,2$ stands for $\mathrm{SU}(2)_R$ doublet indices and $A=1,\ldots,N$ stands for flavours. For simplicity, we assume that these N hypermultiplets have the same U(1) charge, say, unit charge. The U(1) vector multiplet consists of a gauge field W_M, a real scalar field Σ, $\mathrm{SU}(2)_R$ doublet of gauginos λ^i and $\mathrm{SU}(2)_R$ triplet of real auxiliary fields Y^a, where $M,N=0,1,\ldots,4$ denote space-time indices, and $a=1,2,3$ denotes $\mathrm{SU}(2)_R$ triplet index, respectively. In this work, we shall consider a model with the minimal kinetic term for hypermultiplets and vector multiplets. The $\mathcal{N}=2$ SUSY allows only a few parameters in our model: the gauge coupling g, the mass of the A-th hypermultiplet m_A, and the FI parameters ζ^a. The FI parameters are real and transforms as a triplet under $\mathrm{SU}(2)_R$. Then the bosonic part of our lagrangian reads [28]–[32], [38]

$$\begin{aligned}\mathcal{L}_{\text{boson}} = & -\frac{1}{4g^2}(F_{MN}(W))^2+\frac{1}{2g^2}(\partial_M\Sigma)^2+(\mathcal{D}_MH)^\dagger_{iA}(\mathcal{D}^MH^{iA})-H^\dagger_{iA}(\Sigma-m_A)^2H^{iA}+\\ & +\frac{1}{2g^2}(Y^a)^2-\zeta^aY^a+H^\dagger_{iA}(\sigma^aY^a)^i{}_jH^{jA}+F^{\dagger i}_AF^A_i\,,\end{aligned} \tag{2.1}$$

where a sum over repeated indices is understood, $F_{MN}(W)=\partial_MW_N-\partial_NW_M$, covariant derivative is defined as $\mathcal{D}_M=\partial_M+iW_M$, and our metric is $\eta_{MN}=(+1,-1,\ldots,-1)$.

To ensure the discreteness of SUSY vacua, we need to make all hypermultiplet masses nondegenerate. Without loss of generality, we assume the following ordering of the hypermultiplet mass parameters

$$m_{A+1} < m_A \tag{2.2}$$

for all A. When all hypermultiplet masses are nondegenerate, the symmetry of our model reduces to $\mathrm{U}(1)^N$. The diagonal U(1) is gauged by the vector multiplet W_M, and the remaining $\mathrm{U}(1)^{N-1}$ is the global symmetry.

The easiest way to find SUSY vacua is to explore the condition of vanishing vacuum energy. To facilitate the procedure, let us first write down equations of motion of auxiliary fields Y^a and F_i^A

$$Y^a = g^2[\zeta^a - H^\dagger_{iA}(\sigma^a)^i{}_j H^{jA}]\,, \tag{2.3}$$
$$F_i^A = 0\,. \tag{2.4}$$

After eliminating the auxiliary fields, we obtain the potential $V_{\rm pot}$

$$\mathcal{L}_{\rm boson} = -\frac{1}{4g^2}(F_{MN}(W))^2 + \frac{1}{2g^2}(\partial_M\Sigma)^2 + (\mathcal{D}_M H)^\dagger_{iA}(\mathcal{D}^M H^{iA}) - V_{\rm pot}\,, \tag{2.5}$$
$$V_{\rm pot} \equiv \frac{g^2}{2}\left[\zeta^a - H^\dagger_{iA}(\sigma^a)^i{}_j H^{jA}\right]^2 + H^\dagger_{iA}(\Sigma - m_A)^2 H^{iA}\,. \tag{2.6}$$

The vanishing vacuum energy is achieved by requiring

$$H^\dagger_{iB}(\sigma^a)^i{}_j H^{jB} = \zeta^a\,, \tag{2.7}$$

for each $a = 1, 2, 3$ (the flavor index B is summed) and

$$(\Sigma - m_A)H^{iA} = 0\,, \tag{2.8}$$

for each flavor A (the flavor index A is not summed). These SUSY vacuum conditions guarantee the full preservation of SUSY and can also be derived by requiring the SUSY transformation of fermions to vanish as we see immediately.

By making an $\mathrm{SU}(2)_R$ transformation, we can always bring the FI parameters to the third direction without loss of generality

$$\zeta^a = (0, 0, \zeta)\,, \qquad \zeta > 0\,. \tag{2.9}$$

In this choice, we find N discrete SUSY vacua ($A = 1, \ldots, N$) explicitly by solving eqs. (2.8) and (2.7) as

$$\begin{aligned} \Sigma = m_A\,, \qquad |H^{1A}|^2 &= \zeta\,, \quad H^{2A} = 0\,, \\ H^{1B} &= 0\,, \quad H^{2B} = 0\,, \quad (B \neq A)\,. \end{aligned} \tag{2.10}$$

Since fermions are assumed to vanish in the wall configuration, we need to examine only SUSY transformations of fermions to find a configuration preserving a part of SUSY.

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Gaugino λ^i and hyperino ψ^A transforms as[3]

$$\delta_\varepsilon \lambda^i = \left(\frac{1}{2}\gamma^{MN} F_{MN}(W) + \gamma^M \partial_M \Sigma\right)\varepsilon^i + i\left(Y^a \sigma^a\right)^i{}_j \varepsilon^j \,, \tag{2.11}$$

$$\delta_\varepsilon \psi^A = -i\sqrt{2}\left[\gamma^M \mathcal{D}_M H^{iA} + i(\Sigma - m_A) H^{iA}\right]\epsilon_{ij}\varepsilon^j + \sqrt{2} F_i^A \varepsilon^i \,. \tag{2.12}$$

To obtain a wall solution, we assume the configuration to depend only on the coordinate of one extra dimension, which we denote as $y \equiv x^4$. We also assume the four-dimensional Lorentz invariance in the world volume coordinates $x^\mu = (x^0, \ldots, x^3)$, which implies

$$F_{MN}(W) = 0 \,. \tag{2.13}$$

Since we are interested in the $\frac{1}{2}$ BPS configuration, we require the above SUSY transformations (2.12) to vanish for half of the Grassman parameters specified by

$$P_+ \varepsilon^1 = 0 \,, \qquad P_- \varepsilon^2 = 0 \,, \tag{2.14}$$

where $P_\pm \equiv (1 \pm \gamma_5)/2$ are the chiral projection operators. Finally we need to eliminate the auxiliary fields Y^a and F_i^A by their algebraic equations of motion (2.4) to make the BPS condition as first order differential equations for physical fields. Thus we obtain the $\frac{1}{2}$ BPS equations for the massive $\mathcal{N} = 2$ SUSY QED as

$$\partial_y \Sigma = g^2 \left(\zeta - H^\dagger_{1A} H^{1A} + H^\dagger_{2A} H^{2A}\right), \tag{2.15}$$

$$2g^2 H^\dagger_{2A} H^{1A} = 2g^2 H^\dagger_{1A} H^{2A} = 0 \,, \tag{2.16}$$

$$\mathcal{D}_y H^{iA} = (m_A - \Sigma) H^{iA} \,, \qquad i = 1, 2 \,, \quad A = 1, \ldots, N \,. \tag{2.17}$$

One can easily see that the translation invariant vacuum requires the vanishing of the left-hand side of eqs. (2.15) and (2.17), which implies the same condition as the full preservation of SUSY in eqs. (2.8) and (2.7).

We are interested in the solution of these BPS equations which interpolate two different vacua in eq. (2.10). Since the BPS equation is a first order differential equation, the boundary condition dictates that H^{2A} should vanish identically

$$H^{2A}(y) = 0 \,. \tag{2.18}$$

The energy density of the BPS solution can be found by making a Bogomolny completion of the energy functional

$$T_{\rm w} = \int_{-\infty}^{\infty} dy \left\{ \frac{1}{2g^2}\left(\partial_y \Sigma - g^2(\zeta - |H^{1A}|^2)\right)^2 + |\mathcal{D}_y H^{1A} - (m_A - \Sigma) H^{1A}|^2 \right\} + \\ + \left[\zeta \Sigma + (m_A - \Sigma)|H^{1A}|^2\right]_{y=-\infty}^{y=+\infty} . \tag{2.19}$$

[3] Our gamma matrices are 4×4 matrices and are defined as: $\{\gamma^M, \gamma^N\} = 2\eta^{MN}$, $\gamma^{MN} \equiv \frac{1}{2}[\gamma^M, \gamma^N] = \gamma^{[M}\gamma^{N]}$, $\gamma^5 \equiv i\gamma^0\gamma^1\gamma^2\gamma^3 = -i\gamma^4$.

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yielding the BPS wall tension for solutions interpolating the A-th vacuum and the B-th vacuum

$$T_{\rm w} = \zeta(m_A - m_B)\,, \tag{2.20}$$

assuming $A > B$. The structure of the above BPS equations shows that only those hypermultiplet scalars $H^{1C} \neq 0$ with $A \geq C \geq B$ have nonvanishing values besides the vector multiplet scalar Σ [22, 27]. Defining $l \equiv A - B$, we call such a BPS configuration as l wall solutions, since it represents l separate walls at least when these walls are sufficiently far apart, as we shall see in the section 4.

3. Single wall BPS solutions

3.1 Exact solutions for single wall

Since single wall interpolates two adjacent SUSY vacua, we shall assume $N = 2$ (two hypermultiplets), without loss of generality. For simplicity, we take mass parameters of the hypermultiplets as

$$m_1 = -m_2 \equiv m\,. \tag{3.1}$$

The boundary conditions for BPS equations are given by

$$\Sigma(-\infty) = -m\,, \qquad \Sigma(\infty) = m\,, \tag{3.2}$$

$$\begin{aligned} H^{11}(-\infty) &= 0\,, & H^{11}(\infty) &= \sqrt{\zeta}\,, \\ H^{12}(-\infty) &= \sqrt{\zeta}\,, & H^{12}(\infty) &= 0\,. \end{aligned} \tag{3.3}$$

The set of the BPS equations (2.15)–(2.17) with the above boundary conditions (3.2)–(3.3) are known to be solved exactly for infinite gauge coupling [22, 27]. Recently another exact solution has been found for finite gauge coupling, provided the gauge coupling g satisfies the following relation with the ratio of the FI parameter ζ and the mass difference squared $(2m)^2$ [32]

$$g^2\zeta = 2m^2\,. \tag{3.4}$$

By generalizing this exact solution for finite gauge coupling, we find that there are exact solutions for the following values of gauge coupling g

$$g^2\zeta \equiv \frac{8m^2}{k^2}\,, \tag{3.5}$$

with appropriate integers k. We shall denote the single wall solution for k as $S_k(m)$. The integer k indicates the value of the gauge coupling g through the relation (3.5), and the mass parameter m indicates the tension $T_{\rm w}$ of the single wall in unit of the FI parameter as $T_{\rm w} = 2\zeta m$. The infinite gauge coupling corresponds to the case $k = 0$, and the previously obtained finite gauge coupling case to $k = 2$ [32]. We have explicitly obtained exact solutions in the case of $k = 0, 2, 3, 4$.

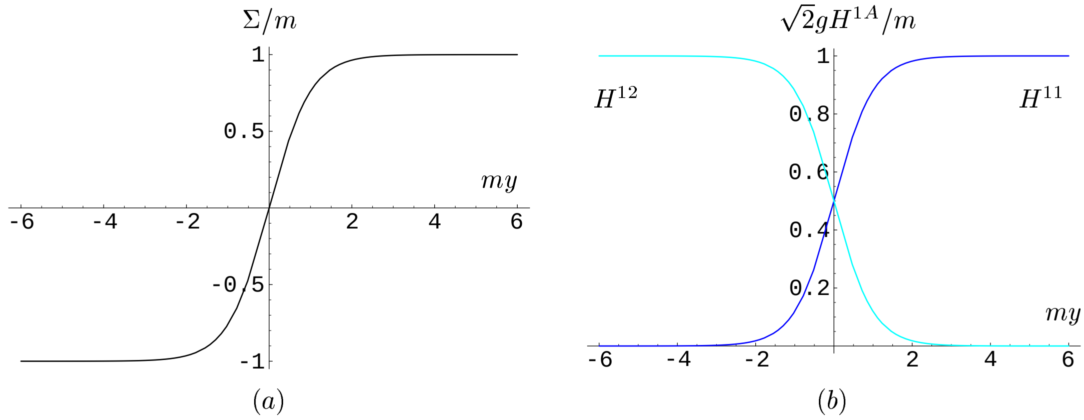

Figure 1: The exact BPS wall solution $S_2(m)$ for finite gauge coupling $g = \sqrt{2}m/\sqrt{\zeta}$ and the tension $T_{\rm w} = 2m\zeta$. (a) Scalar field $\Sigma(y)$ of vector multiplet in eq. (3.6) divided by the mass parameter m as a function of the coordinate y times m. (b) Hypermultiplet scalar field $H^{11}(y)$ in eq. (3.7) as a function of the coordinate y times m (solid line), and $H^{12}(y)$ in eq. (3.8) as a function of the coordinate y times m (dotted line).

In this section, we will describe the exact solution $S_2(m)$ for $k = 2$ as the simplest example, which plays an important role in our paper on massless localized vector field [37]:

$$\Sigma(y) = m \tanh my\,, \tag{3.6}$$

$$H^{11}(y) = \frac{m}{\sqrt{2}g} e^{+my} \,\mathrm{sech}\, my\,, \tag{3.7}$$

$$H^{12}(y) = \frac{m}{\sqrt{2}g} e^{-my} \,\mathrm{sech}\, my\,. \tag{3.8}$$

Let us recall that we can introduce two arbitrary integration constants in these solutions. We can have $y \to y - y_0$ corresponding to the spontaneously broken translation invariance, and we can also have the multiplication by a phase $\mathrm{e}^{-i\alpha}$ ($\mathrm{e}^{i\alpha}$) for hypermultiplets H^{11} (H^{12}) corresponding to the spontaneously broken global U(1) invariance. They constitute two moduli of the solution. The y_0 has a physical meaning of the position of the wall. The vector multiplet scalar Σ and the hypermultiplet scalars H^{11} and H^{12} of the solution $S_2(m)$ are illustrated as a function of the coordinate y in the extra dimension in figure 1. We will present other exact solutions of single wall after introducing a slightly different notation suitable also for multiple walls.

3.2 Fluctuation around the BPS single wall background

Let us now turn our attention to spectra of fluctuations around the BPS single wall background. These fluctuation fields can be expanded into modes defined on the background. Among various modes, we are particularly interested in two kinds of modes: massless modes (zero modes) and fluctuation of vector fields. The former modes play essential role in discussing dynamics on the background at low-energies. The latter is to examine the role played by the dynamical vector field at finite coupling, rather than the role as the Lagrange multiplier field in the case of corresponding nonlinear sigma models.

Zero modes usually arise from several reasons. The first one is the Nambu-Goldstone modes as a result of the spontaneous breaking of continuous global symmetry. The second possibility is the result of remaining symmetry, such as the conserved SUSY [39]. The third possibility is the result of index theorems, such as fermion zero modes around instantons and other solitons [40].

To illustrate the method, we shall work on the fluctuations on the background of the simplest exact solution $S_2(m)$ in eqs. (3.6)–(3.8). In the case of a single wall, such as $S_2(m)$, the only spontaneously broken global internal symmetry is the U(1) symmetry which rotates the phase of hypermultiplets oppositely

$$H^{i1} \to \mathrm{e}^{-i\alpha} H^{i1}\,, \qquad H^{i2} \to \mathrm{e}^{i\alpha} H^{i2}\,. \tag{3.9}$$

There are two bosonic global symmetry which are spontaneously broken: translation and the global U(1) in eq. (3.9). The mode function of the Nambu-Goldstone boson corresponding to the spontaneously broken translation is given by differentiating the background field configuration with respect to the coordinate in extra dimension y. We shall denote the corresponding four-dimensional effective field as $\mathrm{Re}\,\phi_0(x)$. Similarly the Nambu-Goldstone boson associated with the global U(1) symmetry is given by acting an infinitesimal U(1) transformation. We denote the corresponding four-dimensional effective field as $\mathrm{Im}\,\phi_0(x)$. By writing out only zero modes, we thus obtain scalar fields of vector multiplet $\Sigma(x,y)$ and that of hypermultiplets $H^{1A}(x,y), A=1,2$

$$\Sigma(x,y) = \langle\Sigma(y)\rangle - \partial_y\langle\Sigma(y)\rangle\,\mathrm{Re}\,\phi_0(x)\,, \tag{3.10}$$
$$H^{11}(x,y) = \langle H^{11}(y)\rangle - \partial_y\langle H^{11}(y)\rangle\,\mathrm{Re}\,\phi_0(x) - im\langle H^{11}(y)\rangle\,\mathrm{Im}\,\phi_0(x)\,, \tag{3.11}$$
$$H^{12}(x,y) = \langle H^{12}(y)\rangle - \partial_y\langle H^{12}(y)\rangle\,\mathrm{Re}\,\phi_0(x) + im\langle H^{12}(y)\rangle\,\mathrm{Im}\,\phi_0(x)\,, \tag{3.12}$$

where we multiplied the factor m in front of the U(1) Nambu-Goldstone mode wave function for later convenience, and the $\langle\cdots\rangle$ denotes the background field configuration. These configurations together with vanishing gauge field $W_M = 0$ are solutions of the BPS equation if $\mathrm{Re}\,\phi_0$ and $\mathrm{Im}\,\phi_0$ are constants (moduli) and hence automatically satisfy the field equations. However, we are now interested in the case where these $\mathrm{Re}\,\phi_0$ and $\mathrm{Im}\,\phi_0$ are four-dimensional fields, which depend on the coordinate x^μ of the world volume. Then these field configurations are no longer BPS configurations, nor satisfy the field equations. Since four-dimensional effective fields are defined as solutions of the field equations, we should demand these field configurations to satisfy the linearized field equations. Since W_M must vanish for constant ϕ_0, it should be proportional to $\partial_\mu\phi_0(x)$. Therefore the Lorentz invariance in the four-dimensional world volume dictates

$$W_4(x,y) = 0\,, \tag{3.13}$$

and

$$W_\mu(x,y) = w(y)\partial_\mu\,\mathrm{Im}\,\phi_0(x)\,. \tag{3.14}$$

with some function $w(y)$ to the linearized order which we are interested in. Then the linearized field equations for the vector multiplet scalar Σ, hypermultiplet scalars H^{1A} are satisfied by

$$\partial_\mu \partial^\mu \phi_0(x) = 0\,, \tag{3.15}$$

which just means that these Nambu-Goldstone fields are massless.

The remaining linearized field equation for the gauge field W_M reads

$$\partial_\nu \partial^\nu W_\mu - \partial_y^2 W_\mu - \partial_\mu(\partial_\nu W^\nu) + \partial_\mu \partial_y W_4 = -2g^2 \left(\langle H^{1A}\rangle \partial_\mu \operatorname{Im} h^{1A} + |\langle H^{1A}\rangle|^2 W_\mu\right), \tag{3.16}$$

$$\begin{aligned}\partial_\nu \partial^\nu W_4 - \partial_y(\partial_\nu W^\nu) &= 2g^2 \Big(\partial_y \langle H^{1A}\rangle \operatorname{Im} h^{1A} - \langle H^{1A}\rangle \partial_y \operatorname{Im} h^{1A} - \\ &\quad - |\langle H^{1A}\rangle|^2 W_4\Big)\,.\end{aligned} \tag{3.17}$$

The solutions for W_M in eqs. (3.13) and (3.14) are enough to satisfy the field equation for W_4 in eq. (3.17). The remaining field equation (3.16) for W_μ determines the y dependence $w(y)$ of the gauge field $W_\mu(x,y)$ in eq. (3.14) as

$$W_\mu(x,y) = \langle \Sigma(y)\rangle \partial_\mu \operatorname{Im} \phi_0(x)\,. \tag{3.18}$$

This solution shows that the corresponding field strength is nonvanishing around the wall

$$F_{\mu 4}(W)(x,y) = -\partial_y \langle \Sigma(y)\rangle \partial_\mu \operatorname{Im} \phi_0(x)\,. \tag{3.19}$$

In a similar model in four-dimensions, it has been observed that the gauge field must be nontrivial and field strength must be nonzero around the wall if the moduli depends on world volume coordinates [30, 28]. Our result in five dimensions are in agreement with their observation. Our new point is perhaps that the necessary gauge field configuration can be determined explicitly by using the linearized field equations for gauge fields.

SUSY is another global symmetry which is spontaneously broken. We can obtain the Nambu-Goldstone fermion corresponding to the broken SUSY by evaluating the SUSY transformations of fermions on the background. One set of SU(2)-Majorana spinor λ^i, $i = 1, 2$ in five dimensions can be decomposed into two Majorana spinors λ_+, λ_- in four dimensions as

$$\lambda_+ = P_+ \lambda^1 + P_- \lambda^2\,, \qquad \lambda_- = -P_- \lambda^1 + P_+ \lambda^2\,. \tag{3.20}$$

The SUSY transformations in eqs. (2.11) and (2.12) can be evaluated by using the BPS equations (2.15)–(2.17) to yield for the gauginos

$$\begin{aligned}\delta_\varepsilon \lambda_+ &= i\left[\partial_y \langle \Sigma\rangle + g^2\left(\zeta - |\langle H^{1A}\rangle|^2\right)\right](P_+\varepsilon_+ - P_-\varepsilon_+) = 2i\partial_y\langle\Sigma\rangle (P_+\varepsilon_+ - P_-\varepsilon_+) \\ \delta_\varepsilon \lambda_- &= i\left[\partial_y \langle \Sigma\rangle - g^2\left(\zeta - |\langle H^{1A}\rangle|^2\right)\right](P_-\varepsilon_- + P_+\varepsilon_-) = 0\,.\end{aligned} \tag{3.21}$$

Since ε_- is preserved, it drops out from these transformations. If we evaluate explicitly for our exact solution $S_2(m)$ in eqs. (3.6)–(3.8), we obtain

$$\delta_\varepsilon \lambda_+ = \frac{2im^2}{\cosh^2 my}(P_+\varepsilon_+ - P_-\varepsilon_+)\,, \qquad \delta_\varepsilon \lambda_- = 0\,. \tag{3.22}$$

Similarly the hyperino component of the Nambu-Goldstone fermion wave function is given by

$$\begin{aligned}\delta_\varepsilon \psi^A &= \sqrt{2}\left[\partial_y \langle H^{1A}\rangle + (\langle\Sigma\rangle - m_A)\langle H^{1A}\rangle\right] P_+\varepsilon_- + \\ &\quad + \sqrt{2}\left[-\partial_y \langle H^{1A}\rangle + (\langle\Sigma\rangle - m_A)\langle H^{1A}\rangle\right] P_-\varepsilon_+ \\ &= -2\sqrt{2}\partial_y \langle H^{1A}\rangle P_-\varepsilon_+ \,. \end{aligned} \tag{3.23}$$

For the exact solution $S_2(m)$ in eqs. (3.6)–(3.8), it becomes explicitly as

$$\delta_\varepsilon \psi^1 = -\delta_\varepsilon \psi^2 = -\frac{2m^2}{g}\frac{1}{\cosh^2 my} P_-\varepsilon_+ \,. \tag{3.24}$$

The above transformation laws (3.21) and (3.23) show that $-4\varepsilon_+$ is proportional to the zero momentum component of the Nambu-Goldstone fermion. Therefore we can define the Nambu-Goldstone fermion field $\chi_0(x)$ of the four-dimensional effective low-energy theory on the world volume of the wall as

$$\lambda_+(x,y) = -\frac{i}{2}\partial_y \langle\Sigma(y)\rangle \gamma_5 \chi_0(x) \,, \quad \lambda_-(x,y) = 0 \,, \tag{3.25}$$

$$\psi^A(x,y) = \frac{1}{\sqrt{2}}\partial_y \langle H^{1A}(y)\rangle P_- \chi_0(x) \,, \tag{3.26}$$

suppressing to write massive modes. We see explicitly that the Nambu-Goldstone fermion appears only with the left-handed chirality in the hypermultiplets as is usually dictated by index theorems for fermions localized on domain walls [40]. We can also verify that the Nambu-Goldstone fermion satisfies the linearized equations of motion with a vanishing mass eigenvalue, by using the BPS equations (2.15)–(2.17).

Now let us consider the requirement of the preserved symmetry, especially SUSY. The SUSY transformation property under the preserved SUSY ε_- specified by eq. (2.14) is given by

$$\delta_{\varepsilon_-} W_\mu = -i\bar\varepsilon_- \gamma_\mu \lambda_- \,, \tag{3.27}$$

$$\delta_{\varepsilon_-} W_4 = \bar\varepsilon_- \lambda_+ \,, \tag{3.28}$$

$$\delta_{\varepsilon_-} \Sigma = -i\bar\varepsilon_- \gamma_5 \lambda_+ \,, \tag{3.29}$$

$$\delta_{\varepsilon_-} H^{1A} = -\sqrt{2}\bar\varepsilon_- P_- \psi^A \,, \tag{3.30}$$

$$\delta_{\varepsilon_-} H^{2A} = -\sqrt{2}\bar\varepsilon_- P_+ \psi^A \,. \tag{3.31}$$

This transformation property does not fit well with the above massless particles of Nambu-Goldstone modes in Eqs. (3.10)–(3.14), (3.25), and (3.26), especially for the vector multiplet. However, we should remember that we have a freedom to make gauge transformations to make the above massless particles in conformity with the SUSY transformation properties. Let us perform the following gauge transformation, which is proportional to the fluctuation field $\mathrm{Im}\,\phi_0$

$$\begin{aligned} W_4 &\longrightarrow W_4 - \partial_y\left(\langle\Sigma\rangle\,\mathrm{Im}\,\phi_0\right), \\ W_\mu &\longrightarrow W_\mu - \partial_\mu\left(\langle\Sigma\rangle\,\mathrm{Im}\,\phi_0\right), \\ H^{1A} &\longrightarrow \langle H^{1A}\rangle + i\left(\langle\Sigma\rangle\,\mathrm{Im}\,\phi_0\right)\langle H^{1A}\rangle \,. \end{aligned} \tag{3.32}$$

Then the hypermultiplet scalars become

$$\begin{aligned} H^{11}(x,y) &= \langle H^{11}(y)\rangle - \partial_y\langle H^{11}(y)\rangle \operatorname{Re}\phi_0(x) - i\,(m - \langle\Sigma(y)\rangle)\,\langle H^{11}(y)\rangle \operatorname{Im}\phi_0(x) \\ &= \langle H^{11}(y)\rangle - \partial_y\langle H^{11}(y)\rangle\,(\operatorname{Re}\phi_0(x) + i\operatorname{Im}\phi_0(x))\,, \qquad (3.33) \\ H^{12}(x,y) &= \langle H^{12}(y)\rangle - \partial_y\langle H^{12}(y)\rangle \operatorname{Re}\phi_0(x) - i\,(-m - \langle\Sigma(y)\rangle)\,\langle H^{12}(y)\rangle \operatorname{Im}\phi_0(x) \\ &= \langle H^{12}(y)\rangle - \partial_y\langle H^{12}(y)\rangle\,(\operatorname{Re}\phi_0(x) + i\operatorname{Im}\phi_0(x))\,. \qquad (3.34) \end{aligned}$$

Therefore we find that the two Nambu-Goldstone bosons corresponding to translation and U(1) global rotation forms a complex scalar $\phi_0(x)$ of a chiral scalar multiplet of the preserved four SUSY ϵ_-. At the same time, the four-dimensional gauge fields W_μ vanishes, and vector multiplet scalar Σ and the fourth (extra dimension) component of vector field W_4 fit into the complex scalar $\phi_0(x)$

$$\begin{aligned} \Sigma(x,y) + iW_4(x,y) &= \langle\Sigma(y)\rangle - \partial_y\langle\Sigma(y)\rangle\,(\operatorname{Re}\phi_0(x) + i\operatorname{Im}\phi_0(x))\,, \qquad (3.35) \\ W_\mu(x,y) &= 0\,. \qquad (3.36) \end{aligned}$$

These mode expansions together with the fermionic ones in eqs. (3.25) and (3.26) are now consistent with the preserved SUSY which defines the four-dimensional $\mathcal{N}=1$ SUSY transformations for the massless fields in the effective field theory

$$\delta_{\varepsilon_-}\phi_0(x) = \bar{\varepsilon}_- P_- \chi_0(x)\,. \qquad (3.37)$$

Thus we find that all the Nambu-Goldstone bosons and fermions together form a chiral scalar multiplet $(\phi_0(x), \chi_0(x))$ of the preserved SUSY. Moreover we do not find any index theorem to force us additional zero modes. Therefore we do not expect any more zero modes.

Let us now turn to the issue of mass spectra, especially for vector fields. The linearized equation of motion for vector field W_μ and W_4 are given in eqs. (3.16) and (3.17). By exploiting the U(1) gauge invariance, we can always choose a gauge where $W_4(x,y) = 0$ identically. We can also decompose vector field into the transverse part $\tilde{W}_\mu$ and longitudinal part w

$$W_\mu(x,y) = \tilde{W}_\mu(x,y) + \partial_\mu w(x,y)\,, \qquad \partial^\mu \tilde{W}_\mu = 0\,. \qquad (3.38)$$

The transverse part of the vector field[4] is decoupled from the rest of the fluctuations because of the four-dimensional Lorentz invarince and gives the following mode decomposison with the mode function $a_n(y)$ for the mass eigenvalue m_n of the four-dimensional effective field $W_\mu^{(n)}(x)$

$$\begin{aligned} \tilde{W}_\mu(x,y) &= \sum_n a_n(y) W_\mu^{(n)}(x)\,, \qquad (3.39) \\ -\partial_y^2 a_n(y) + V(y) a_n(y) &= m_n^2 a_n(y)\,. \qquad (3.40) \end{aligned}$$

where the potential $V(y)$ is defined by

$$V(y) = 2g^2 |\langle H^{1A}\rangle|^2\,, \qquad (3.41)$$

[4]This component corresponds to the polazation states proportional to the polazation vector ϵ^i_μ, $i = 1,2,3$ defined by $\epsilon^i_\mu p^\mu = 0$ in momentum space, representing a genuine four-dimensional vector field.

and is illustrated in figure 2 for the background $S_2(m)$ in eqs. (3.6)–(3.8), and $S_3(m)$ and $S_4(m)$, whose exact solutions are given explicitly in section 4.1. These potentials have a dip near the wall and approaches $2g^2\zeta$ at $y \to \pm\infty$. Therefore all the vector fluctuations become infinitely heavy as $g \to \infty$. Potentials are always positive definite, but have attractive forces due to a dip around the wall for any finite gauge coupling. It is likely that there may be discrete low mass states before reaching a continuum starting from $m^2 = 2g^2\zeta$. The potential becomes infinite for the infinite gauge coupling. It is clear that the eigenvalue can never vanish: $m_n > 0$. It is also easy to see that m_n are of the order of m for, say, $S_2(m)$. This result is consistent with the general argument that the gauge field will obtain a mass of the order of the inverse width of the wall. Although the low-lying vector modes have presumably discrete spectra and are localized around the wall, all these modes are massive. To obtain a massless vector field, we can add tensor multiplet [37].

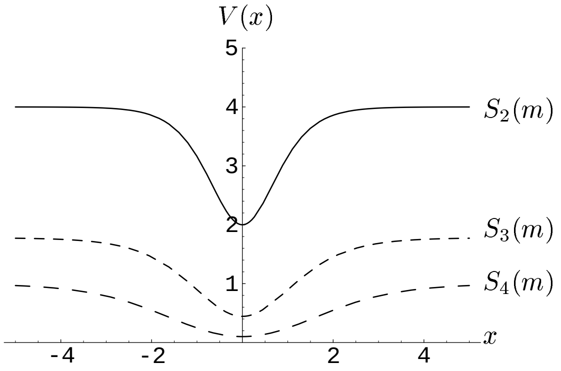

Figure 2: The potential $V(x)$ of eq. (3.41) for $S_2(m)$ (solid line), $S_3(m)$ (dotted line), and $S_4(m)$ (dashed line) in the eigenvalue equation for the vector fluctuation (3.40) as a function of extra dimension coordinate y. The mass parameter is taken to be $m = 1$.

To find out the remaining field equations, we define fluctuations of field as $\Sigma = \langle\Sigma\rangle + s$, $H^{Ai} = \langle H^{Ai}\rangle + h^{Ai}$. The linearized field equations are given by

$$0 = \partial_M \partial^M s + 2g^2 \langle |H^{Ai}|^2 \rangle s + 2g^2 h^\dagger_{Ai} \langle (\Sigma - m_A) H^{Ai} \rangle + 2g^2 \langle H^\dagger_{iA} (\Sigma - m_A) \rangle h^{Ai} , \quad (3.42)$$

$$\begin{aligned} 0 = {} & \partial_M \partial^M h^{Ai} + i(\partial^M W_M) \langle H^{Ai} \rangle + 2i W_M \partial^M \langle H^{Ai} \rangle - g^2 (\zeta - \langle |H^{Bj}|^2 \rangle) h^{Ai} + \quad (3.43) \\ & + g^2 \langle H^{Ai} \rangle \left(\langle H^\dagger_{Bj} \rangle h^{Bj} + \langle H^{jB} \rangle h^\dagger_{Bj} \right) + 2(\langle\Sigma\rangle - m_A) \langle H^{Ai} \rangle s + (\langle\Sigma\rangle - m_A)^2 h^{Ai} . \end{aligned}$$

These linearized field equations can be decomposed into three sets of coupled equations. The first set consists of the fluctuations of vector multiplet scalar Σ, and those of the real part of the hypermultiplet scalars $\mathrm{Re}\, H^{1A}$, $A = 1, 2$. The second set consists of imaginary part of the hypermultiplet scalars $\mathrm{Im}\, H^{1A}$, $A = 1, 2$ and the longitudinal part of the vector field w in eq. (3.38). The third set consists of the lower components of the hypermultiplet scalars H^{2A}, $A = 1, 2$. Among various massive modes, we find the following tower of massive modes as a solution of the first set of coupled equations

$$s = \sum_n (\partial_y a_n(y)) \phi_n(x) , \qquad h^{A1} = -\sum_n \langle H^{A1}(y) \rangle a_n(y) \phi_n(x) , \quad (3.44)$$

with $h^{A2} = W_M = 0$. This effective field $\phi_n(x)$ has the identical mass squared m_n^2 as the effective field $W_\mu^{(n)}(x)$ of the transverse vector in eq. (3.40). Moreover the mode function is also given by the same wave function $a_n(y)$ as the transverse vector. Therefore it is likely that these effective fields are related by the preserved symmetry, in particular the $\mathcal{N} = 1$ SUSY. In fact, a massive vector multiplet of $\mathcal{N} = 1$ SUSY should contain a scalar particle.

4. BPS solution for multiple walls

4.1 Exact solutions with finite gauge couplings

It is convenient to introduce a complex function $\psi(y)$ to solve the BPS equation for hypermultiplets (2.17) [28]

$$H^{1A}(y) = \sqrt{\zeta} \exp\left(-\psi(y) + m_A(y - y_0) + \sum_{a=1}^{N-2} \alpha_A{}^a r_a\right), \tag{4.1}$$

where the complex parameters $r_a, a = 1, \ldots, N-2$ are collective coordinates arising as integration constants. Since two complex parameters among the integration constants can be absorbed by a shift of ψ and y_0, we can choose the $N \times (N-2)$ fixed real matrix $\alpha_A{}^a$ to be of rank $N-2$. The BPS equation for hypermultiplets is equivalent to the following equation for $\psi(y)$

$$\partial_y \psi = \Sigma + i W_4 \,. \tag{4.2}$$

Because of the vanishing field strength (2.13), the vector field $W_4 = \mathrm{Im}\, \partial_y \psi$ is a pure gauge. However, when we consider dynamics of domain walls, this term will play an important role.

We have the following moduli parameters in the solution (4.2),

$$y_0 = Y_0 + i\theta_0 \,, \tag{4.3}$$

$$r_a = R_a + i\theta_a \,, \tag{4.4}$$

with one and $N-2$ complex dimensions, respectively. The Y_0 and R_a are related to the center of mass and relative positions of the a-th domain wall, respectively, and θ_0 and θ_a to the overall phase and relative phases of the a-th wall, respectively. Using the variable ψ, the BPS equation for vector multiplet (2.15) becomes

$$\begin{aligned} \frac{1}{\zeta g^2} \partial_y^2 \mathrm{Re}(\psi) &= 1 - \sum_{A=1}^{N} \exp\left(-2\,\mathrm{Re}(\psi) + 2m_A(y - Y_0) + 2\sum_{a=1}^{N-2} \alpha_A{}^a R_a\right) \\ &= 1 - \exp\left(-2\,\mathrm{Re}(\psi) + 2W\right). \end{aligned} \tag{4.5}$$

Here, the explicit dependence on y in eq. (4.5) can be assembled by defining a function $W(y)$ as

$$W = \log \sum_{A=1}^{N} \exp\left(2m_A(y - Y_0) + 2\sum_{a=1}^{N-2} \alpha_A{}^a R_a\right). \tag{4.6}$$

To ensure that $\partial_y \Sigma = 0$ at $y \to \pm\infty$, we should impose boundary conditions for the above equation as

$$\mathrm{Re}(\psi) \longrightarrow m_1(y - Y_0) + \sum_{a=1}^{N-2} \alpha_1{}^a R_a \,, \qquad y \to +\infty \,, \tag{4.7}$$

$$\mathrm{Re}(\psi) \longrightarrow m_N(y - Y_0) + \sum_{a=1}^{N-2} \alpha_N{}^a R_a \,, \qquad y \to -\infty \,. \tag{4.8}$$

To define an appropriate variable for the position of each wall, let us consider a configuration where only two adjacent exponential terms in the sum of eq. (4.5) are large and the others are negligible, then the function W has a profile for a single wall. Therefore it is natural to define the position y_A of the A-th wall as

$$\exp\left(2m_A(y_A-Y_0)+2\sum_{a=1}^{N-2}\alpha_A{}^aR_a\right)=\exp\left(2m_{A+1}(y_A-Y_0)+2\sum_{a=1}^{N-2}\alpha_{A+1}{}^aR_a\right)$$
$$\to\quad y_A=Y_0-\frac{\sum_{a=1}^{N-2}\left(\alpha_{A+1}{}^a-\alpha_A{}^a\right)R_a}{m_{A+1}-m_A}\,. \qquad (4.9)$$

This moduli parametrization y_A has an intuitive meaning of the position of the A-th wall and y_A-y_{A+1} corresponds to a distance between two walls, at least when the distance is large. Relations between y_A and the relative positions R_a defined in eq. (4.4) are obtained by

$$-\sum_{a=1}^{N-2}(\alpha_A{}^a-\alpha_1{}^a)R_a=\sum_{B=1}^{A-1}\left(m_{B+1}-m_B\right)\left(y_B-Y_0\right),\qquad A=2,\ldots,N\,. \qquad (4.10)$$

By choosing $A=N$, we find

$$Y_0=\frac{1}{m_N-m_1}\left[\sum_{B=1}^{N-1}\left(m_{B+1}-m_B\right)y_B+\sum_{a=1}^{N-2}(\alpha_N^a-\alpha_1^a)R_a\right], \qquad (4.11)$$

which becomes the center of mass coordinate when $\alpha_N^a=\alpha_1^a$. Because of the translational invariance, $\mathrm{Re}(\psi)$ should be a function of real variables $y-Y_0$ and R_a. From now on, we will take $Y_0=0$ unless otherwise stated.

Now we will present a series of exact solutions for single and double walls for finite values of gauge coupling. To compare single and multi-wall solutions, let us assign the following mass parameters for hypermultiplets for double walls ($N=3$):

$$m^A=(m,0,-m)\,. \qquad (4.12)$$

This convention is intended to make the total energy density of the double wall to be identical to the single wall, so that the double wall situation can be most naturally compared to the single wall situation with the same energy density (tension) in the coincident limit of two walls, since the mass parameters of the single wall is assigned to be $m^A=(m,-m)$ in eq. (3.1), and the total energy density is just given by the difference of the two extreme masses as given in eq. (2.20). For the moduli parameter, we use $y_1-y_2=R^1\equiv R$ and choose the $\alpha_A{}^1$ as

$$\alpha_A{}^1=\left(0,\frac{m}{2},0\right),\qquad 2\alpha_A{}^aR_a=2\alpha_A{}^1R_1=(0,mR,0)\,, \qquad (4.13)$$

since the rank of the matrix $\alpha_A{}^a$ is $N-2$. This relative distance appears only in multiple wall, but not in the single wall.

The function $W\equiv W_{\mathrm{single}}$ for the single wall case becomes

$$W_{\mathrm{single}}\equiv\frac{1}{2}\log(\mathrm{e}^{2my}+\mathrm{e}^{-2my})\,. \qquad (4.14)$$

Similarly the function for the double wall case $W\equiv W_{\mathrm{double}}$ is found to be

$$W_{\mathrm{double}}\equiv\frac{1}{2}\log(\mathrm{e}^{2my}+\mathrm{e}^{-2my}+\mathrm{e}^{mR}). \qquad (4.15)$$

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The solvable cases of finite gauge coupling are found to be

$$g^2\zeta \equiv \frac{8m^2}{k^2}\,, \qquad k = 0, 2, 3, 4\,, \tag{4.16}$$

where the mass parameter m is defined for single wall case in eq. (3.1), and double wall case in eq. (4.12). We will denote the single wall solution as $S_k(m)$ and the double wall solution as $D_k(m)$ with the coupling defined by (4.16) with k and the mass parameter m.

Let us list the exact solutions that we are able to obtain

$$S_0(m):\ \ \mathrm{Re}\,\psi = \frac{1}{2}\log(\mathrm{e}^{2my} + \mathrm{e}^{-2my})\,, \tag{4.17}$$

$$S_2(m):\ \ \mathrm{Re}\,\psi = \log(\mathrm{e}^{my} + \mathrm{e}^{-my})\,, \tag{4.18}$$

$$S_3(m):\ \ \mathrm{Re}\,\psi = \frac{3}{2}\log(\mathrm{e}^{\frac{2}{3}my} + \mathrm{e}^{-\frac{2}{3}my})\,, \tag{4.19}$$

$$S_4(m):\ \ \mathrm{Re}\,\psi = \log(\mathrm{e}^{my} + \mathrm{e}^{-my} + \sqrt{6})\,, \tag{4.20}$$

$$D_0(m):\ \ \mathrm{Re}\,\psi = \frac{1}{2}\log(\mathrm{e}^{2my} + \mathrm{e}^{-2my} + \mathrm{e}^{mR})\,, \tag{4.21}$$

$$D_4(m):\ \ \mathrm{Re}\,\psi = \log(\mathrm{e}^{my} + \mathrm{e}^{-my} + \sqrt{6 + \mathrm{e}^{mR}})\,. \tag{4.22}$$

It is interesting to observe that the double wall solution $D_0(m)$ in eq. (4.21) in the case of infinite coupling can be rewritten as a superposition of two single walls placed apart by a distance $\tilde{R}$

$$\mathrm{Re}\,\psi(y) = \frac{1}{2}\log\left(\mathrm{e}^{m(y-\frac{\tilde{R}}{2})} + \mathrm{e}^{-m(y-\frac{\tilde{R}}{2})}\right) + \\ +\frac{1}{2}\log\left(\mathrm{e}^{m(y+\frac{\tilde{R}}{2})} + \mathrm{e}^{-m(y+\frac{\tilde{R}}{2})}\right). \tag{4.23}$$

This parameter $\tilde{R}$ can be regarded as another choice of a moduli parameter and is related to R defined in eq. (4.4)

$$\mathrm{e}^{m\tilde{R}} + \mathrm{e}^{-m\tilde{R}} = \mathrm{e}^{mR} \tag{4.24}$$

$$\Rightarrow \quad m\tilde{R} = \log\left(\frac{\mathrm{e}^{mR} + \sqrt{\mathrm{e}^{2mR} - 4}}{2}\right),$$

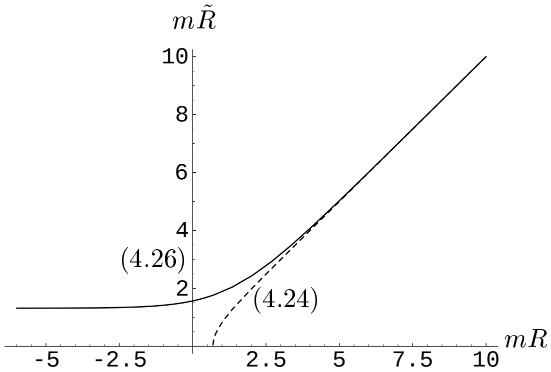


Figure 3: Relation between R defined in eq. (4.4) and $\tilde{R}$. The solid line represents the relation defined in eq. (4.26) for the finite coupling solution $D_4(m)$ in eq. (4.22). The dashed line represents the relation defined in eq. (4.24) for the infinite coupling solution $D_0(m)$ in eq. (4.21).

which is illustrated in figure 3. Unfortunately the new choice of the moduli parameter $\tilde{R}$ becomes pure imaginary and loses an intuitive meaning as the distance between the two walls when $R < \log 2/m$. This situation has some similarity to a moduli parameter for a model of BPS double wall in an $\mathcal{N} = 1$ SUSY Wess-Zumino model in four-dimensions [41]. We can also rewrite the double wall solution $D_4(m)$ in eq. (4.22) in the case of the finite coupling as a superposition of two single walls placed apart by a distance $\tilde{R}$

$$\mathrm{Re}\,\psi(y) = \log\left(\mathrm{e}^{\frac{m}{2}(y-\frac{\tilde{R}}{2})} + \mathrm{e}^{-\frac{m}{2}(y-\frac{\tilde{R}}{2})}\right) + \log\left(\mathrm{e}^{\frac{m}{2}(y+\frac{\tilde{R}}{2})} + \mathrm{e}^{-\frac{m}{2}(y+\frac{\tilde{R}}{2})}\right). \tag{4.25}$$

The new moduli parameter $\tilde{R}$ is related to R in this case as

$$\mathrm{e}^{\frac{m\tilde{R}}{2}}+\mathrm{e}^{-\frac{m\tilde{R}}{2}}=\sqrt{6+\mathrm{e}^{mR}} \quad \Rightarrow \quad m\tilde{R}=2\log\left(\frac{\sqrt{6+\mathrm{e}^{mR}}+\sqrt{2+\mathrm{e}^{mR}}}{2}\right). \tag{4.26}$$

In this case of the finite coupling $D_4(m)$ in eq. (4.22), the parameter $\tilde{R}$ takes only positive real values, whereas R takes positive as well as negative values as illustrated in figure 3. In both cases of infinite and finite coupling, both R and $\tilde{R}$ have an intuitive meaning of relative distance between the wall, as long as the distance is large: $\tilde{R}\to R$, for $R\to\infty$. Therefore $\tilde{R}$ in this case gives an intuitively nicer parametrization for the relative distance between the two walls.

It is interesting to examine how these double wall solutions behave, in the limit of coincident walls and in the limit of asymptotically far apart walls. By smoothly changing the moduli parameter R, we find the following limiting behaviors which can be expressed symbolically as

$$S_0(m) \overset{R\to-\infty}{\longleftarrow} D_0(m) \overset{R\to+\infty}{\longrightarrow} S_0\left(\frac{m}{2}\right)\oplus S_0\left(\frac{m}{2}\right), \tag{4.27}$$

$$S_4(m) \overset{R\to-\infty}{\longleftarrow} D_4(m) \overset{R\to+\infty}{\longrightarrow} S_2\left(\frac{m}{2}\right)\oplus S_2\left(\frac{m}{2}\right). \tag{4.28}$$

These limiting behaviors apply not only for the $\psi(y)$, but also for all the physical quantities. Note that the sum of the numbers k is preserved, corresponding to the fact that the total tension is a conserved topological invariant for a given boundary condition. We can find Σ, H^{1A} and the potential V corresponding to these exact single and double wall solutions (4.17)–(4.22) by using (4.1), (4.2) and (3.41). We illustrate and compare various single and double wall solutions for scalar $\Sigma(y)$ of vector multiplet in figure 4, for hypermultiplet scalars $H^{1A}(y)$ for $A=1,2$ (single wall) and $A=1,2,3$ (double wall) in figure 5 and for potential $V(y)$ in figure 6.

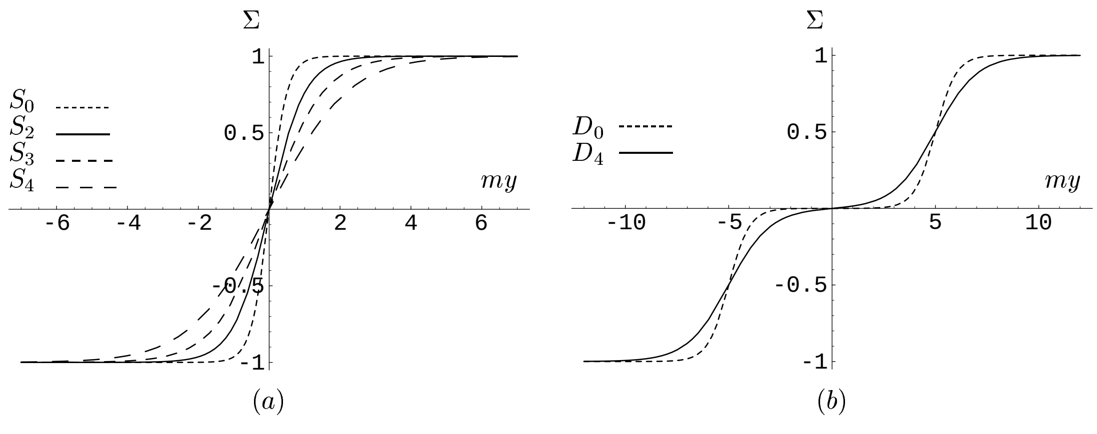


Figure 4: Comparison of the vector multiplet scalar Σ as a function of my for exact solutions with various gauge couplings. (a) Single wall solutions $S_0(m)$ (dotted line), $S_2(m)$ (solid line), $S_3(m)$ (short dashed line), and $S_4(m)$ (dashed line). (b) Double wall solutions $D_0(m)$ (dotted line), and $D_4(m)$ (solid line).

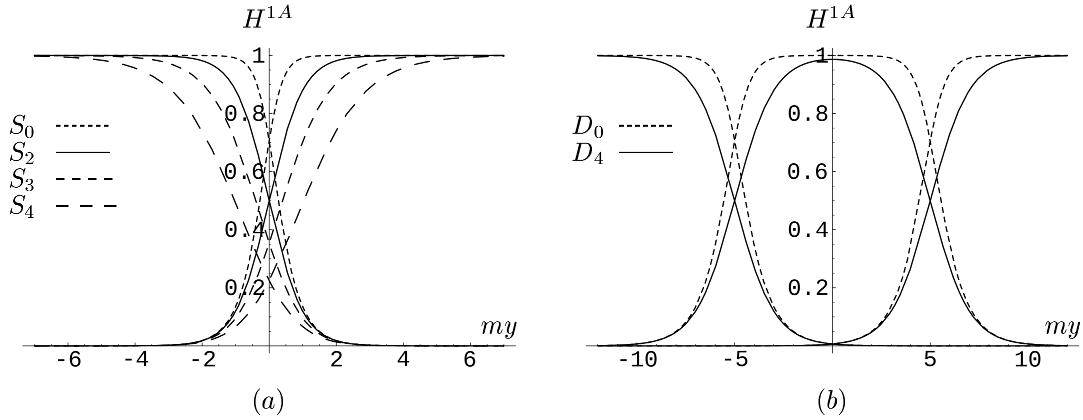


Figure 5: Comparison of the hypermultiplet scalars H^{1A} as a function of my for exact solutions with various gauge couplings. a) Single wall solutions $S_0(m)$ (dotted line), $S_2(m)$ (solid line), $S_3(m)$ (short dashed line), and $S_4(m)$ (dashed line). b) Double wall solutions $D_0(m)$ (dotted line), and $D_4(m)$ (solid line).

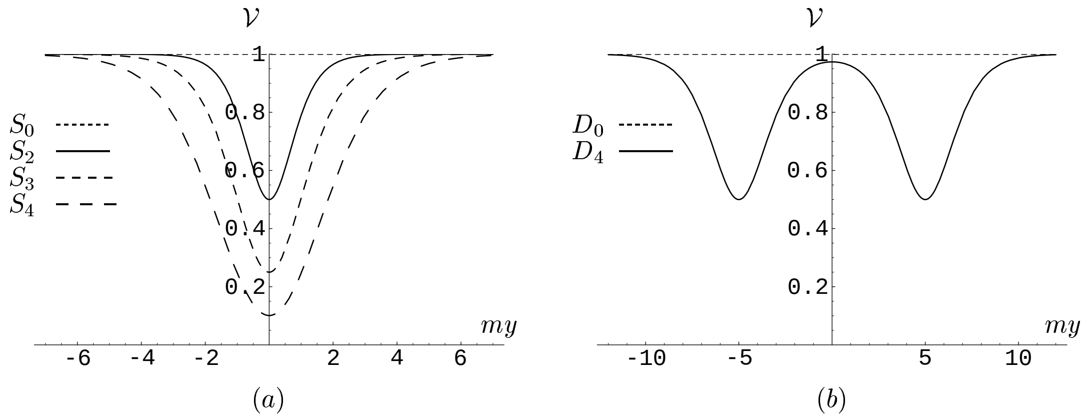


Figure 6: Comparison of potential $\mathcal{V} \equiv V/2g^2\zeta$ normalized by the asymptotic value ($y \to \pm\infty$) as a function of my for exact solutions with various gauge couplings. (a) Single wall solutions $S_0(m)$ (dotted line), $S_2(m)$ (solid line), $S_3(m)$ (short dashed line), and $S_4(m)$ (dashed line). (b) Double wall solutions $D_0(m)$ (dotted line), and $D_4(m)$ (solid line).

4.2 Moduli dynamics of two domain walls

The low-energy effective lagrangian for massless field has been studied systematically, especially in the case of infinite gauge coupling [28]. It has been proposed to use an expansion in inverse powers of gauge coupling, since the infinite gauge coupling case was the only explicitly known solution at that time. We would like to derive the low-energy effective lagrangian for massless field in the case of finite gauge coupling, using our exact solutions. We will compare it to the previous result of infinite gauge coupling, and show an instability of an expansion in inverse powers of gauge coupling.

Since the overall position and phase become just free massless fields [28], we shall concentrate on the relative distance R_a and relative phase θ_a of multiple walls defined in eq. (4.4). These moduli parameters represent positions in y and in internal space of U(1) of multiple walls. We have obtained mode functions of massless Nambu-Goldstone fields by differentiating the background in terms of these moduli parameters in section 3.2. Here we are interested in not only the mass spectrum but also the entire effective lagrangian at low energies. To find out the low-energy effective lagrangian, a more systematic method by Manton [33] is useful. In this method, we need to promote the parameters of the solution (moduli) to be functions of the world volume coordinates x^μ, namely four-dimensional fields. Assuming the variation in the world volume coordinates x^μ to be weak, we can obtain all the nonlinear dynamics containing smallest number of derivatives (two derivatives). This well-defined procedure allows us to obtain the nonlinear kinetic term of the massless fields. Since we are considering the massless Nambu-Goldstone fields, there should be no additional potential terms without derivatives. We can keep only $t \equiv x^0$ dependence of these fields, since the Lorentz invariance in the world volume allows to recover the entire dependence on x^μ [28]. Effective lagrangian $\mathcal{L}_{\rm eff}$ of moduli fields is found to be given by integrating over the lagrangian of the fundamental theory evaluated by the background field with the t dependent moduli $r_a(t) = R_a(t) + i\theta_a(t)$ [28]

$$\mathcal{L}_{\rm eff} = \int_{-\infty}^{\infty} dy \sum_{A=1}^{N} |H^{1A}|^2 (\dot{\psi} + m_A \dot{y}_0 - \alpha_A^a \dot{r}^a)(m_A \dot{y}_0^\dagger - \alpha_A^a \dot{r}_a^\dagger) \,. \tag{4.29}$$

Let us take two wall case ($N = 3$) for concreteness. We discard terms for the overall position and phase, and omit the subscript a, since there is only one set of relative moduli R, θ now. We will be interested in low-energy effective lagrangian for the relative distance R and the relative phase θ of two walls. To obtain the dependence on the relative phase θ, we need to find the imaginary part of $\psi(y)$ besides the real part that we have determined. The BPS equation (4.2) shows that the imaginary part of $\psi(y)$ is given by integrating the extra dimension component of the gauge field: $\partial_y \operatorname{Im} \psi(y) = W_4(y)$. As shown in the case of the single wall in section 3.2, we have, near the walls, a nontrivial field strength $F_{\mu y}(W) \neq 0$ proportional to the derivative of the moduli fields $\dot{\theta}$. Then the extra dimension component of the vector potential is determined by the field equation for the vector potential [28]. The most convenient gauge to make the remaining SUSY manifest is $W_\mu = 0$ and $W_4 \neq 0$.

We find from the equation of motion that the imaginary part (combined with the real part) satisfies

$$\frac{\partial^2 \dot{\psi}}{\partial y^2} = 2g^2 \zeta \left(\dot{\psi} - \dot{r} \frac{\partial W(r + r^*)}{\partial r} \right) e^{-\psi - \psi^* - 2W} \,. \tag{4.30}$$

Because of the boundary condition $\psi = 0$ at $y = \pm\infty$, the solutions of the real and imaginary parts are proportional to $\dot{R}$ and $\dot{\theta}$ with the same coefficient which is a function of R only

$$\operatorname{Re} \dot{\psi} = \dot{R} \frac{\partial}{\partial R} \operatorname{Re} \psi(y, R) \,, \qquad \operatorname{Im} \dot{\psi} = \dot{\theta} \frac{\partial}{\partial R} \operatorname{Re} \psi(y, R) \,, \tag{4.31}$$

implying $\dot{W}_4 = \partial_y \frac{\partial}{\partial R} \operatorname{Re} \psi(y, R) \dot{\theta}$. The moduli field $r \equiv R + i\theta$ is a complex scalar of a chiral scalar field, taking values [28] $R \in \mathbb{R}$ and $\theta \in [0, 4\pi/m)$. We see that the $\mathcal{N} = 1$ SUSY in four dimensions is now manifest. Moreover, the Kähler metric $\mathcal{K}_{rr^*}$ of the low-energy effective lagrangian $\mathcal{L}_{\rm eff}$ is given in terms of $F(R)$ which is a function of R only

$$\mathcal{L}_{\rm eff} = \left(\dot{R}^2+\dot{\theta}^2\right)\mathcal{K}_{rr^*}(r,r^*)\,, \tag{4.32}$$

$$\mathcal{K}_{rr^*}(r,r^*) = \frac{1}{4}m\zeta F(mR) = \frac{1}{4}m\zeta F\left(m\frac{r+r^*}{2}\right). \tag{4.33}$$

The function $F(R)$ is most conveniently evaluated by taking $\theta(t) = 0$ and by writing hypermultiplets H^{1A} in terms of the function $\psi(y)$ in eq. (4.1) and using the choice α_A in eq. (4.13)

$$\begin{aligned}\frac{1}{4}m\zeta\dot{R}^2F(mR) &= \int_{-\infty}^{\infty}dy\sum_{A=1}^{3}|H^{1A}|^2(-\operatorname{Re}\dot{\psi}\alpha_A\dot{R}+\alpha_A^2\dot{R}^2)\\ &= \frac{1}{4}m\zeta\dot{R}^2\frac{d}{dR}\int_{-\infty}^{\infty}dy{\rm e}^{-2\psi(y)+mR}\,.\end{aligned} \tag{4.34}$$

The Kähler metric $\mathcal{K}_{rr^*}$ gives equations of motion for the relative distance R and for the relative phase θ

$$\ddot{R} = -\frac{1}{2}m(\log F)'(\dot{R}^2-\dot{\theta}^2)\,, \tag{4.35}$$

$$\ddot{\theta} = -m(\log F)'\dot{R}\dot{\theta}\,, \tag{4.36}$$

respectively, where $'$ means $\partial/\partial z$, $z \equiv mR$. These equations of motion are real and imaginary parts of a single equation due to the complex structure of the $\mathcal{N}=1$ SUSY:

$$\ddot{r} = -\frac{1}{2}m(\log F)'\dot{r}^2\,, \qquad r\equiv R+i\theta\,. \tag{4.37}$$

We observe that the above equations of motion admit that $\dot{\theta}=0$ is always a consistent solution. Therefore we can discuss the relative motion of the double wall with fixed relative phase consistently. On the other hand, in order to have a fixed R, the above equations of motion is consistent only if θ is also constant. Therefore we shall consider the relative motion assuming a constant relative phase θ in the following.

Inserting our exact wall solution $D_4(m)$ with the finite gauge coupling in eq. (4.22) into eq. (4.34), we obtain the corresponding Kähler metric $F(z=mR)$ for the finite gauge coupling

$$\begin{aligned}F(z)_4 &= {\rm e}^z\int_{-\infty}^{\infty}dt\frac{{\rm e}^t+{\rm e}^{-t}+\frac{6}{\sqrt{6+{\rm e}^z}}}{({\rm e}^t+{\rm e}^{-t}+\sqrt{6+{\rm e}^z})^3}\\ &= \frac{{\rm e}^z}{(2+{\rm e}^z)^2}\left\{{\rm e}^z-4+\frac{24}{\sqrt{(2+{\rm e}^z)(6+{\rm e}^z)}}\log\left[\frac{\sqrt{6+{\rm e}^z}+\sqrt{2+{\rm e}^z}}{2}\right]\right\}.\end{aligned} \tag{4.38}$$

For the $D_0(m)$ solution at the infinite coupling, the Kähler metric F has been obtained as [28]

$$\begin{aligned}F(z)_0 &= \frac{{\rm e}^z}{2}\int_{-\infty}^{\infty}dt\frac{{\rm e}^t+{\rm e}^{-t}}{({\rm e}^t+{\rm e}^{-t}+{\rm e}^z)^2}\\ &= \frac{{\rm e}^z}{{\rm e}^{2z}-4}\left\{{\rm e}^z+\frac{4}{\sqrt{{\rm e}^{2z}-4}}\log\left[\frac{2}{{\rm e}^z+\sqrt{{\rm e}^{2z}-4}}\right]\right\}.\end{aligned} \tag{4.39}$$

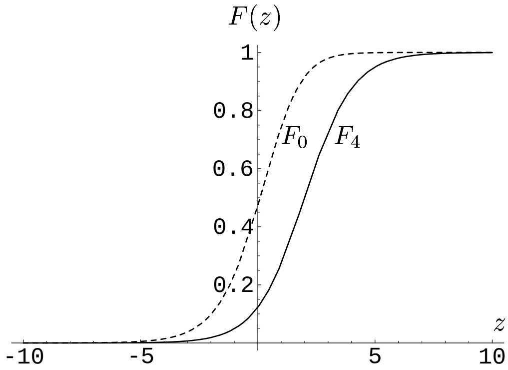


Figure 7: Comparison of Kähler metrics $F(z)_0$ for the infinite gauge coupling (dashed line) and $F(z)_4$ for the finite gauge coupling (solid line), as functions of $z = mR$.

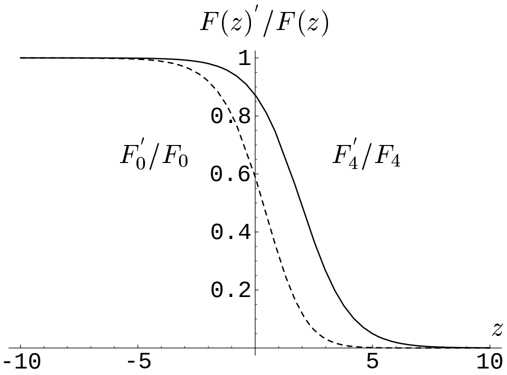


Figure 8: Comparison of the ratio F_0'/F_0 (dashed line) and F_4'/F_4 (solid line).

The metric is real and positive for the entire values of $-\infty < z < \infty$, in spite of the apparent singularity at $\mathrm{e}^{2z} = 4$ [28]. These Kähler metrics are illustrated and compared in figure 7. We see that there is a significant difference between them quantitatively, although general features are similar.

The physical significance of the quantitative difference can perhaps most easily be appreciated by comparing the strengths of forces acting between the walls in the case of the finite coupling to that of the infinite coupling. In figure 8, we illustrate and compare the coefficients F'/F of force in eqs. (4.35)–(4.37) for the infinite and finite coupling cases. We observe that the strength of the force for the finite coupling case is much larger than the infinite coupling case for large relative distance R.

To understand the quantitative difference of the strength of the forces at large values of R, let us evaluate the Kähler metric F at asymptotic values of R. In the region of $R \to -\infty$, both target space metrics $F(z)$ are flat and are related each other by a scale transformation. On the other hand, in the case of $R \to +\infty$, our lagrangian for the finite coupling becomes

$$\mathcal{L}_4 = \frac{1}{4} m\zeta \left[1 - 8\mathrm{e}^{-mR} + (12mR + 28)\mathrm{e}^{-2mR}\right](\dot{R}^2 + \dot{\theta}^2)\,. \tag{4.40}$$

Consequently the equation of motion of the relative distance R for $D_4(m)$ solution at the finite gauge coupling is found to be

$$\ddot{R} = \left[-\mathrm{e}^{-mR} 4m + \mathrm{e}^{-2mR}(12m^2 R - 10m)\right](\dot{R}^2 - \dot{\theta}^2). \tag{4.41}$$

On the other hand, the asymptotic lagrangian for infinite coupling at $R \to \infty$ has been found to be [28]

$$\mathcal{L}_0 = \frac{1}{4} m\zeta \left[1 - 4(mR - 1)\mathrm{e}^{-2mR}\right](\dot{R}^2 + \dot{\theta}^2), \tag{4.42}$$

and the corresponding $R \to \infty$ limit of the equation of motion of the relative distance R for $D_0(m)$ solution at the infinite coupling reads [28]

$$\ddot{R} = -\mathrm{e}^{-2mR}(4m^2 R - 6m)(\dot{R}^2 - \dot{\theta}^2). \tag{4.43}$$

We see that for $\dot{R}^2 > \dot{\theta}^2$ there is always an attractive force operating between walls which are sufficiently far apart, irrespective of the strength of the gauge coupling. However, we find that the strength of the force for the finite coupling behaves as e^{-mR}, which is much larger than that for the infinite coupling case of e^{-2mR}. This explains the reason why we obtained much stronger force between the walls in the case of the finite coupling.

4.3 Approximation in $1/g^2$ and asymptotic behaviors

Here we wish to discuss the previously proposed expansion in inverse powers of gauge coupling to obtain finite coupling results [28]. We also analyze the asymptotic behavior of the BPS wall solutions for generic values of gauge coupling to understand the reason why the power series approximation exhibits pathological behavior.

We expand $\mathrm{Re}\,\psi$ in inverse powers of the gauge coupling: $a \equiv (2g^2\zeta)^{-1} = k^2/(4m)^2$ as

$$\mathrm{Re}\,\psi = \sum_{n=0}^{\infty} a^n \psi_n \,. \tag{4.44}$$

By substituting the expansion to the equation (4.5), we obtain an expansion in power of a. The leading power $1/a$ comes only from the right-hand side and gives the result at the infinite coupling in eq. (4.21): $\psi_0 = W$. Using this result, all the successive powers can be solved iteratively

$$\sum_{n=0}^{\infty} a^n \psi_{n+1} = \sum_{n=0}^{\infty} a^n \frac{d^2\psi_n}{dy^2} - 2\sum_{n=1}^{\infty} \frac{(-2)^n a^n}{(n+1)!} \left(\sum_{l=0}^{\infty} a^l \psi_{l+1} \right)^{n+1} \,. \tag{4.45}$$

Several lower order results are given explicitly by

$$\begin{aligned} \psi_0 &= W \,, \qquad \psi_1 = W^{(2)} \,, \qquad \psi_2 = W^{(4)} + (W^{(2)})^2 \,, \\ \psi_3 &= W^{(6)} + 4W^{(4)}W^{(2)} + 2(W^{(3)})^2 + \frac{4}{3}(W^{(2)})^3 \,, \\ \psi_4 &= W^{(8)} + 6W^{(6)}W^{(2)} + 12W^{(5)}W^{(3)} + 9(W^{(4)})^2 + 12W^{(4)}(W^{(2)})^2 + \\ &\quad + 12(W^{(3)})^2W^{(2)} + 4(W^{(2)})^4, \; \ldots, \end{aligned} \tag{4.46}$$

where $W^{(n)}$ is defined by $W^{(n)} \equiv d^nW/dy^n$.

First let us compare the power series approximation with our exact solution of the single wall $S_2(m)$ for the finite gauge coupling $k = 2$. We illustrate the result of the power series expansion up to the l-th order $\Sigma^{(l)}(y) = \partial_y\psi^{(l)}(y) \equiv \sum_{n=0}^{l} a^n \partial_y \psi_n(y)$ by setting $a \equiv (2g^2\zeta)^{-1} = (2m)^{-2}$, and compare it with the exact solution $\partial_y\psi(y)$ in figure 9. Near the wall, the approximation oscillates wildly and shows no indication of convergence.

Similarly we can obtain the power series approximations for the Kähler metric F for the relative position and phase for two walls. For concreteness, we will take the case of our exact solution $D_4(m)$ for the finite coupling $k = 4$, and compare the exact Kähler metric $F(z)_4$ with a profile obtained by the approximation in inverse powers of the gauge coupling. We can obtain the approximations up to the l-th order $F(z)_4^{(l)}$, by inserting the above expansion (4.46) to the action (4.34) and by setting $a = (2g^2\zeta)^{-1} = m^{-2}$. In

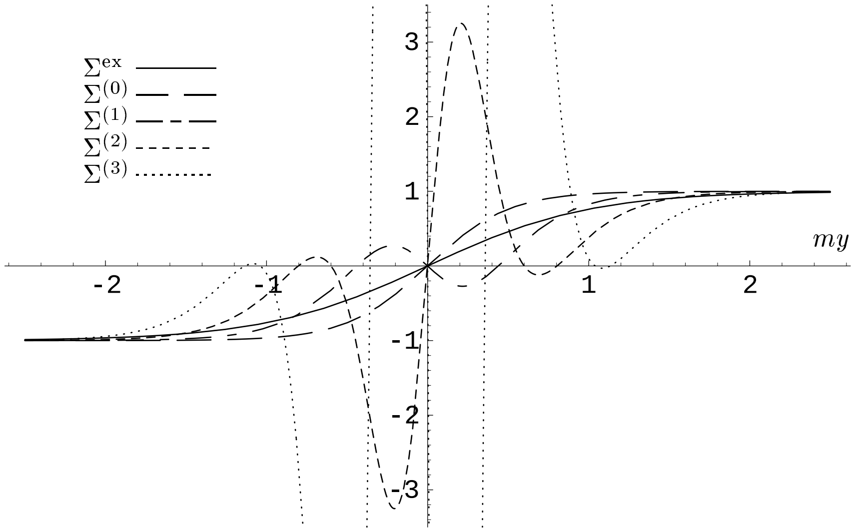
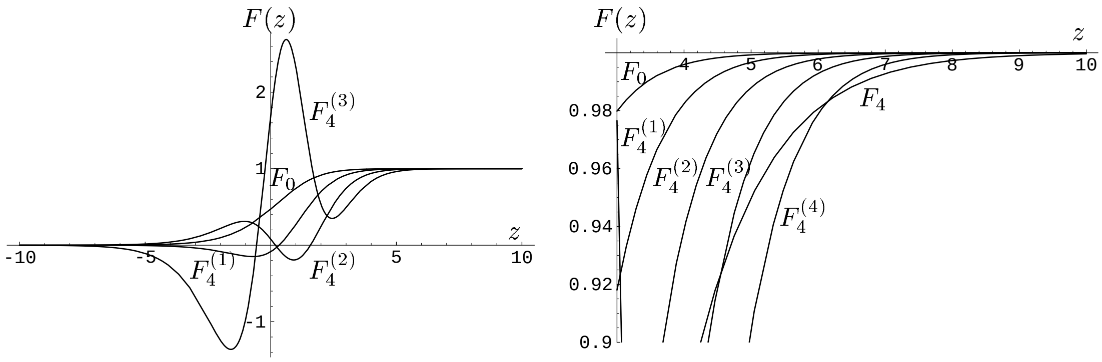


Figure 9: Comparison of approximations in inverse powers of gauge coupling to the exact solution $\Sigma^{\rm ex}(y)$ for the vector multiplet scalar $\Sigma(y) = \partial_y \,{\rm Re}\,\psi(y)$ with a finite gauge coupling $g^2\zeta = 2m^2$. Approximation up to the l-th order is denoted as $\Sigma^{(l)}$. The zero-th order approximation (infinite gauge coupling) $\Sigma^{(0)}$ is represented by a dashed line. $\Sigma^{(1)}$ by a dash-dotted line, $\Sigma^{(2)}$ by a short dashed line, and $\Sigma^{(3)}$ by a dotted line. The mass parameter is set to $m = 1$.



(a) (b)

Figure 10: Comparison of approximations in inverse powers of gauge coupling up to the l-th order $F_4^{(l)}(z)$ to the exact solution $F_4(z)$ for the Kähler metric of moduli fields as a function of $z \equiv mR$. (a) Near the region of coincident walls. (b) Asymptotic region $y = mR \to \infty$.

figure 10a, we compare the power series approximation up to various orders starting from the zero-th order F_0 (infinite gauge coupling case) up to the third order approximations $F_4^{(3)}$. We see wild oscillations near the wall. As the order of approximation increases, this oscillatory behavior becomes wilder, without any indication of dying out. If we abandon to describe the region near the wall, we can examine the approximation in the asymptotic regions away from the wall. We also illustrate the Kähler metric $F(z)$ for larger values of $z = mR \to \infty$ in figure 10b. Even there, there is no indication for the series to converge to our exact result. Actually, we can make the discrepancy more quantitative. For instance let us take the power series approximation up to the fourth order $F(mR)_k^{(4)}$ for the function

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$F(mR)$ with an arbitrary coupling $a = (2g^2\zeta)^{-1} = (k/(4m))^2$. Its asymptotic behavior at $R \to \infty$ is given by

$$
\begin{aligned}
F(mR)_k^{(4)} &= 1 + \Bigg(-4(1 + 4m^2 a + 16m^4 a^2 + 64m^6 a^3 + 256m^8 a^4) mR + \\
&\qquad\qquad + 4 + 24m^2 a + 128m^4 a^2 + \frac{2176}{3} m^6 a^3 + \frac{58112}{15} m^8 a^4 + O(a^5) \Bigg) e^{-2mR} + \\
&\quad + O(e^{-4mR}) \,,
\end{aligned}
\tag{4.47}
$$

whereas the exact form of the function with the finite coupling at $R \to \infty$ is

$$
F(mR)_4 = 1 - 8e^{-mR} + (12mR + 28) e^{-2mR} + O(e^{-4mR}) \,. \tag{4.48}
$$

It is now clear that the power series approximation $F(mR)_4^{(l)}$, $l \to \infty$ can never converge to the function $F(mR)_4$. Therefore we conclude that the approximation by an expansion in inverse powers of the gauge coupling is pathological, and may not be suitable to extract physical quantities correctly.

Finally we analyze the asymptotic behavior of the BPS solution for arbitrary finite gauge coupling. This can be worked out even without exact solutions and may provide an alternative approximation scheme, rather than the expansion in inverse powers of gauge coupling. Let us take single wall case with two hypermultiplets for simplicity. Since BPS solutions are essentially real when the moduli parameters are constants (not fields), we can ignore $\mathrm{Im}\, \psi(y)$ and obtain the BPS equation as

$$
\frac{d^2\psi}{dy^2} = g^2\zeta \left(1 - e^{-2\psi} (e^{2my} + e^{-2my}) \right) . \tag{4.49}
$$

Let us parametrize the gauge coupling in terms of k which is no longer an integer here: $g^2\zeta = 8m^2/k^2$. We wish to devise an approximation scheme valid for large values of $z \equiv my$. Since the vector multiplet scalar $\Sigma = \partial_y \psi(y)$ should approach to a constant asymptotically, the BPS equation dictates that $\psi(y)$ should approach $\pm my$ at $y \to \pm\infty$. We expand the remaining subleading order terms in a series which will be determined successively as solutions of iterative equations

$$
\psi(y) = my + \frac{1}{2} \sum_{n=1}^{\infty} \varepsilon_n(y) \,, \tag{4.50}
$$

where we can assume that the remaining terms should vanish asymptotically, since we can absorb a possible constant term into a shift of y

$$
\varepsilon_n(y) \to 0 \quad \text{as } y \to 0, \qquad \varepsilon_1(y) \gg \varepsilon_2(y) \gg \cdots . \tag{4.51}
$$

By substituting the expansion to the BPS equation, we obtain

$$
\frac{1}{2} \sum_{n=1}^{\infty} \frac{d^2 \varepsilon_n}{dy^2} = g^2\zeta \left(1 - e^{-\sum_{n=1}^{\infty} \varepsilon_n} (1 + e^{-4my}) \right) . \tag{4.52}
$$

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We can rewrite this equation with $z \equiv my$ and $g^2\zeta = 8m^2/k^2$

$$\sum_{n=1}^{\infty} \frac{d^2\varepsilon_n}{dz^2} = \frac{16}{k^2}\left(1 - e^{-\sum_{n=1}^{\infty}\varepsilon_n}(1 + e^{-4z})\right),$$
$$= \frac{16}{k^2}\left(\varepsilon_1 + \varepsilon_2 - \frac{1}{2}\varepsilon_1^2 + \cdots - e^{-4z}(1 - \varepsilon_1 + \cdots)\right). \qquad (4.53)$$

The first order approximation is determined by the following equation

$$\frac{d^2\varepsilon_1}{dz^2} = \frac{16}{k^2}\left(\varepsilon_1 - e^{-4z}\right), \qquad (4.54)$$

which yields a general solution with an arbitrary integration constant c_1

$$\varepsilon_1 = 2c_1 e^{-\frac{4z}{k}} + \frac{1}{1-k^2}e^{-4z}, \qquad (4.55)$$

where we have used the boundary condition $\varepsilon_1 \to 0$ at $y \to \infty$ to eliminate another possible integration constant. Therefore the asymptotic behavior of the first order approximation $\varepsilon_1(y)$ depends on the strength of the gauge coupling constant $g^2 = 8m^2/k^2\zeta$ as

$$\varepsilon_1 = \begin{cases} 2c_1 e^{-\frac{4z}{k}} : & k > 1,\ c_1 > 0 \\ 2(z - z_0)e^{-4z} : & k = 1 \\ \frac{1}{1-k^2}e^{-4z} : & 0 < k < 1\,. \end{cases} \qquad (4.56)$$

The wall solution requires that the vacuum value m is reached from below: $\Sigma(y) = \partial_y\psi(y) < m$ as $y \to \infty$, This implies $\varepsilon_1(y) > 0$. Therefore we obtain up to the first order approximation

$$\psi \to \begin{cases} my + c_1 e^{-\frac{4}{k}my} : & k > 1 \\ my + m(y - y_0)e^{-4my} : & k = 1 \\ my + \frac{1}{1-k^2}e^{-4my} : & 0 < k < 1 \end{cases} \qquad (4.57)$$

This result can be regarded as the origin of a different asymptotic behavior for the finite coupling exact solution ($k = 2$ case) compared to the infinite coupling solution ($k = 0$ case). It is interesting to observe that the critical case of $k = 1$ has an exceptional asymptotic behavior corresponding to degenerate exponents. This complication may be related to the fact that we have not yet succeeded to obtain an exact solution in that case.

It is now straightforward to extend our approximation scheme to more general multi-wall cases. Since our approximation scheme from asymptotic region is applicable to any values of gauge coupling constant, this should be useful as an alternative approximation method instead of the expansion in inverse powers of gauge coupling. Our iterative approximation is at least guaranteed to be valid for asymptotic region $y \to \infty$, although it may wildly oscillate and possibly diverge near the wall. On the other hand, we have seen that the expansion in power of inverse gauge coupling does not converge near the wall nor asymptotically. To obtain a full solution, one needs to determine the integration constant such as c_1 which can be done by smoothly connecting to solutions from the region near the wall. Only when this integration constant is chosen to be a particular value, the solution should smoothly connect to the solution near the wall.

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